Mock modularity of CY threefolds

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October 2024

Based on paper with S.Alexandrov to appear

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  - Compute the entropy of black holes.
  - It can give insight into quantum corrections to the Bekenstein-Hawking entropy formula.
  - They also appear as weights for instanton contributions to the low energy effective action.
- These topological invariants can be assembled into functions that posess remarkable modular properties.

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- Modularity specifies how a function (modular form) transforms when the modular group $SL(2,\mathbb{Z})$ acts on its (complex) argument.
- Modular forms obey very rigid constraints.
- We can use these constraints to find the (generating function) of the topological invariants exactly!

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Result

We use these modular properties to fix, up to ambiguities, these generating functions.

Modular forms

- The modular group $SL(2,\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} |\, \mathrm{Im}\,\tau > 0\}$ through

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- A modular form $f(\tau) : \mathbb{H} \to \mathbb{C}$ of weight k is holomorphic and transforms as

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- We see modularity implies $f(\tau + 1) = f(\tau)$, which implies that f has a Fourier expansion

$$f(\tau) = \sum_{n\geq 0} c_n \mathrm{q}^n, \qquad \mathrm{q} = e^{2\pi i\tau}.$$

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- For a given k, modular forms of weight k form a finite dimensional vector space.
- Modularity gives good control on the growth of the Fourier coefficients c_n.
- We will present three generalizations to modular forms that appear, together, in our work.

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- In the limit $z \to 0$ we recover a modular form.
- Remark: compared to the more known Jacobi forms, this definition misses the elliptic transformation of z.

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- Mock modular functions with a given weight k and a given shadow g form a finite dimensional space.

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where $\rho=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathcal{M}_{\mu\nu}(\rho)$ is a representation of the modular group. We call $\mathcal{M}_{\mu\nu}$ the multiplier system.

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- The space of such forms, for given k and $\mathcal{M}$, is finite dimensional.
- Our generating functions are vector-valued.

Modularity cheat sheet

Term	Math. object	Characterestics
Modular form	$f(\tau)$	weight k
VV Modular form	$f_\mu(\tau)$	multiplier system $\mathcal{M}_{\mu\nu}$
Jacobi-like form	$f_\mu(\tau, z), f_\mu(\tau, z_1, z_2)$	index m; indices m_1, m_2
Mock modular form	$f(\tau) \leftrightarrow \hat{f}(\tau, \bar{\tau})$	shadow $g(\tau)$

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where the role of $c_{n,\mu}$ is played by the invariants.
- We will fix f_μ up to computing a finite number of $c_{n,\mu}$.

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- We denote them $\Omega(\gamma)$ and they take integer values.
- Physically:
  - They count the number of microstates of black holes with charge γ of type IIA string theory compactified on $\mathfrak{Y}$.
  - They appear as weights of instanton contributions to the low energy effective theory coming from type IIB string theory on $\mathfrak{Y}$ [S. Alexandrov, KB '23].

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- This allows us to define a (vector-valued) generating function for each magnetic charge p

$$h_{p,\mu}(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\max}} \bar{\Omega}_{p,\mu}(\hat{q}_0)\, \mathrm{q}^{-\hat{q}_0},$$

where $\mathrm{q} = e^{2\pi i\tau}$.

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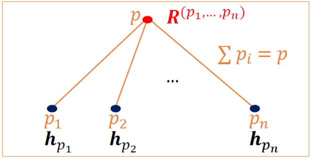
$$\widehat{h}_{p,\mu}(\tau,\bar{\tau}) = \sum_{n=1}^{p} \sum_{\sum_{i=1}^{n} p_i = p} \sum_{\{\mu_i\}} R^{\{p_i\}}_{\mu,\{\mu_i\}}(\tau_2) \prod_{i=1}^{n} h_{p_i,\mu_i}(\tau),$$

where $\tau_2 = \mathrm{Im}\,\tau$.



The completion equation of $h_{p,\mu}$

- Example for $p = 2$

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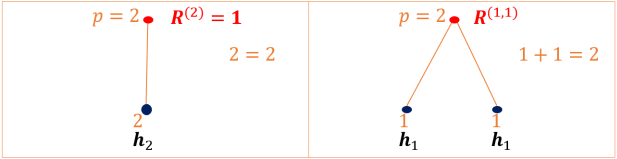

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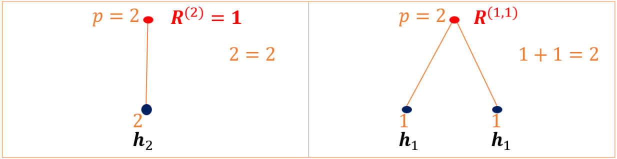

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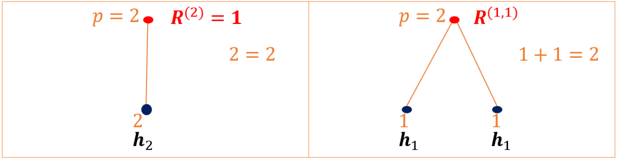

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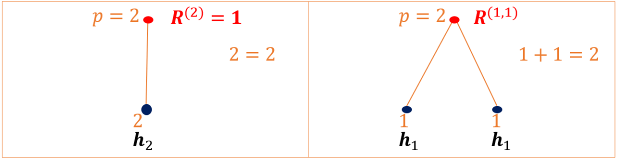

- This equation doesn't characterise $h_{2,\mu}$ completely! Given one solution, we can add any modular holomorphic function to it and get another solution.
- We can fix the ambiguity by computing a few DT invariants.
- This suggests a two-step approach to finding $h_{p,\mu}$.

The two steps

- We decompose $h_{p,\mu} = h^{(an)}_{p,\mu} + h^{(0)}_{p,\mu}$ where $h^{(an)}_{p,\mu}$ is a particular solution to the equation and $h^{(0)}_{p,\mu}$ is the holomorphic modular ambiguity [S.Alexandrov, N.Gaddam, J.Manschot, B.Pioline '22].

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  - Find a particular solution $h_p^{(an)}$.
  - Compute a finite number of DT invariants and fix $h_p^{(0)}$.
- Problem: How to perform the first step for all p without also performing the second step for all $p_i < p$? Because the completion equation of h_p depends on all the holomorphic modular ambiguities of lower charges.

Strategy

- We provide an ansatz to extract the dependance of the generating functions on lower rank ambiguities

$$h_{p,\mu}(\tau) = \sum_{n=1}^{p} \sum_{\sum_{i=1}^{n} p_i = p} \sum_{\{\mu_i\}} g_{\mu,\{\mu_i\}}^{\{p_i\}}(\tau) \prod_{i=1}^{n} h_{p_i,\mu_i}^{(0)}.$$

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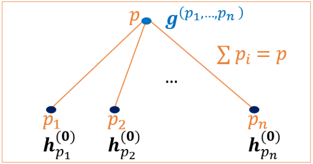

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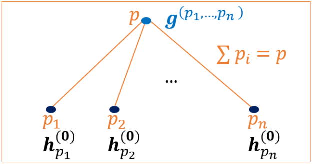
$$h_{p,\mu}(\tau) = \sum_{n=1}^{p} \sum_{\sum_{i=1}^{n} p_i = p} \sum_{\{\mu_i\}} g_{\mu,\{\mu_i\}}^{\{p_i\}}(\tau) \prod_{i=1}^{n} h_{p_i,\mu_i}^{(0)}.$$



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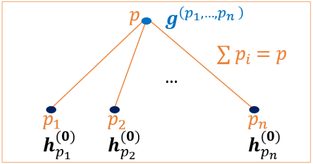

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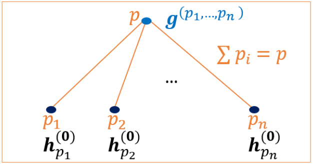

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Goal

Find the *anomalous coefficients*.

Anomalous coefficients

- These functions are mock modular of depth $n-1$ and their completion is given by:

$$\widehat{g}^{\{p_i\}} = \mathrm{Sym}\left\{ \sum_{\sum_i n_i = n} \mathrm{R}^{\{s_i\}}(\tau_2) \prod_{i=1}^{k} g^{(p_{j_i+1},\ldots,p_{j_{i+1}})}(\tau) \right\},$$

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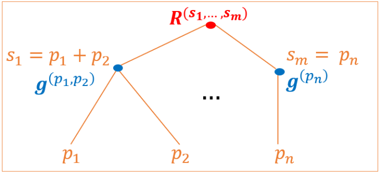

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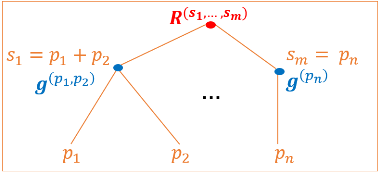

- The main blocks, $\mathrm{R}^{\{r_i\}}(\tau_2)$, are non-holomorphic **theta series**.

Theta series

- A simple theta series can be written as

$$\vartheta_\mu = \sum_{k\in\Lambda+\mu} \mathrm{q}^{-\frac{1}{2}Q(k)^2},$$

where Λ is a d-dimensional lattice with negative definite quadratic form $Q(x)$ that verifies $Q(x) \in 2\mathbb{Z}$. It gives a vector-valued modular form with the dimension of the representation being equal to $|\det Q|$.

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- If $Q(x)$ is indefinite, we can still define a theta series by inserting a kernel $\Phi(\sqrt{2\tau_2}\,k)$ that has support inside the negative cone of the quadratic form.

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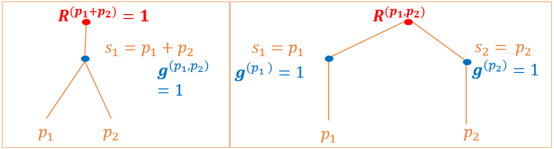
- There are 2 possibilities, either we take a (product of) difference of sign functions which preserves holomorphicity but spoils modularity, or we take the kernel as (product of) difference of generalized error functions which ensures modularity but spoils holomorphicity.

Solving the completion equation

- Let's look at the equation at $n = 2$

$$\widehat{g}^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}(\tau,\bar{\tau}) = g^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}(\tau) + \mathrm{R}^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}(\tau_2)$$



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- This suggests we should choose $g^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}(\tau)$ as a theta series such that the sum of its kernel with that of $\mathrm{R}^{\{p_i\}}_{\mu,\{\mu_i\}}(\tau_2)$ is a solution of the Vignéras equation.

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- The kernel that accomplishes this, while ensuring holomorphicity, is constructed using sign functions: $(\mathrm{sgn}(v \cdot k) - \mathrm{sgn}(w \cdot k))$ where $w \in \Lambda$ and is null (i.e $Q(w) = 0$) and v is fixed by $\mathrm{R}^{\{p_i\}}_{\mu,\{\mu_i\}}(\tau_2)$.

Solving the completion equation

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- Our lattice is of definite signature and doesn't contain null vectors $\implies$ we need to extend the lattice.
- There is another step that we need to do before writing the solution: adding a refinement parameter [S. Alexandrov, J. Manschot, B. Pioline '20].

The refinement

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- We recover the original functions when $z \to 0$.

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- A solution to this new equation descends to a solution of the refined equation through

$$g^{\{p_i\}\mathrm{ref}}_{\mu,\{\mu_i\}}(\tau, z) = \left[\prod_{i=1}^{n} \mathcal{D}^{(\kappa p_i)}_{z_i}\, \tilde{g}^{\{p_i\}\mathrm{ref}}_{\mu,\{\mu_i\}}(\tau, z, \{z_i\})\right]\Bigg|_{\{z_i \to 0\}},$$

where $\mathcal{D}^{(\kappa p_i)}_{z_i}$ are modular derivatives acting on the extra refinements parameters z_i introduced with the extension.

The solution for $n = 2$

- For the $n = 2$ case we examined earlier, a solution reads:

$$\tilde{g}^{(p_1,p_2)\mathrm{ref}}_{\mu,\mu_1,\mu_2} = \vartheta^{(p_1,p_2)}_{\mu,\mu_1,\mu_2} + \phi^{(p_1,p_2)}_{\mu,\mu_1,\mu_2},$$

where ϑ is an idefinite theta series with kernel $(\mathrm{sgn}(v \cdot k) - \mathrm{sgn}(w \cdot k + \beta))$ with $w \in \tilde{\Lambda}$.

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- The second function ϕ is a holomorphic modular ambiguity that cancels the pole in z and ensures a regular limit $z \to 0$.

The solution for any n

- We provide a solution to the refined extended completion equation in the form

$$\tilde{g}^{\{p_i\}\mathrm{ref}}_{\mu,\{\mu_i\}} = \mathrm{Sym}\Bigg\{ \sum_{\sum n_i = n} \vartheta^{\{s_i\}}_{\mu,\{\nu_i\}} \prod_{k=1}^{m} \phi^{\{\mathscr{R}_k\}}_{\nu_k,\{\mu_j\}_{j_k < j \leq j_{k+1}}} \Bigg\},$$

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which mimics the form of the completion equation verified by $\tilde{g}^{\{p_i\}\mathrm{ref}}_{\mu,\{\mu_i\}}$.

- There are two parts to this solution, the indefinite theta series $\vartheta^{\{p_i\}}_{\mu,\{\mu_i\}}$ and the Jacobi-like forms $\phi^{\{p_i\}}_{\mu,\{\mu_i\}}$

The indefinite theta series

- Each $\vartheta^{\{p_i\}}_{\mu,\{\mu_i\}}$ is an indefinite theta series with the kernel $\prod_{i=1}^{n} \left(\text{sgn}(v_i \cdot k) - \text{sgn}(w_i \cdot k + \beta)\right)$

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- The presence of β in the sign functions regularizes the sum over directions $w_i \cdot k = 0$. These regularized directions produce poles in $z = 0$.

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- They also ensure that the solution $\tilde{g}^{\mathrm{ref}\{p_i\}}$ has a regular unrefined limit $z \to 0$.
- They can be chosen

$$\phi^{\{p_i\}}_{\mu,\{\mu_i\}}(\tau, z) \propto \delta^{(\kappa p_0)}_{\mu-\sum_i \mu_i} \frac{e^{-\frac{m}{3}\pi^2 E_2(\tau) z^2}}{z^{n-1}},$$

where m is the index of the full function and $E_2(\tau)$ is the (second) Eisenstein series.

Results

- This recipe allows to find an explicit expression for the anomalous coefficients $g^{\{p_i\}}_{\mu,\{\mu_i\}}(\tau)$ for any number of charges $p_1,\ldots,p_n$.

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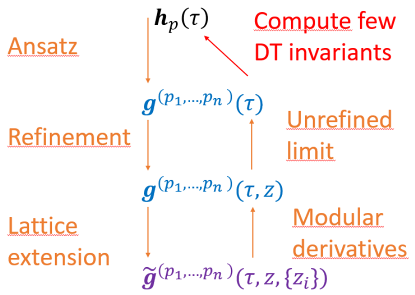
- The anomalous coefficients were found explicitly, in full generality for 2 and 3 charges.
- We tested our solutions against known solutions for charges $(1,1,1)$ and a few examples with two charges (r_1, r_2).
- In principle we can go to higher number of charges and thus find a particular solution $h_p^{(an)}$ up to fixing all modular ambiguities $h_{p_i}^{(0)}$ for $p_i < p$.

- We parametrized the dependence of $h_{p,\mu}$ on $h^{(0)}_{p_i,\mu}$ with $p_i \leq p$ through $g^{\{p_i\}}_{\mu,\{\mu_i\}}$.



- This opens up various development directions:
  - Compute polar terms to fix the $h^{(0)}_{p,\mu}$. (Done for $p=2$ for two CY [S.Alexandrov, S.Feyzbakhsh, A.Klemm '23])
  - Generalize the construction for $b_2 > 1$.

A. Comparing solutions

- If we have two solutions $g^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}$ and $\mathfrak{g}^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}$ then the combination

$$\varphi^{(p_1,p_2)}(\tau,z) = \sum_{\mu,\mu_i} \left(g^{(p_1,p_2)\mathrm{ref}}_{\mu,\mu_1,\mu_2} - \mathfrak{g}^{(p_1,p_2)\mathrm{ref}}_{\mu,\mu_1,\mu_2} \right) \vartheta^{(p_1,p_2)}_{\mu,\mu_1,\mu_2},$$

  is a Jacobi form with known weight and index.
- One can decompose it in a basis of the space of Jacobi forms of that given weight and index.

A. Explicit solution

The solution we find for charges $(1,1)$ reads:

$$g_0^{(1,1)} = \frac{7}{497664\,q} - \frac{7573}{82944} - \frac{11993\,q}{3456} - \frac{6147187\,q^2}{15552} - \frac{417892013\,q^3}{20736} - \frac{2669990303\,q^4}{4608} + O\left(q^5\right)$$

$$g_1^{(1,1)} = \frac{247}{62208\,q^{1/4}} + \frac{2441\,q^{3/4}}{2592} - \frac{685847\,q^{7/4}}{6912} - \frac{60354863\,q^{11/4}}{7776} - \frac{1794183169\,q^{15/4}}{6912} + O\left(q^{19/4}\right)$$

B. Eisenstein series

The expression of $E_2(\tau)$ is

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)\, \mathrm{q}^n,$$

where $\sigma_1(n) = \sum_{d|n} d$. It transforms as

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\left(E_2(\tau) + \frac{6}{\mathrm{i}\pi}\,\frac{c}{c\tau+d}\right).$$